

# Finite-time fluctuations in the degree statistics of growing networks

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**Abstract** This paper presents a comprehensive analysis of the degree statistics in models for growing networks where new nodes enter one at a time and attach to one earlier node according to a stochastic rule. The models with uniform attachment, linear attachment (the Barabási-Albert model), and generalized preferential attachment with initial attractiveness are successively considered. The main emphasis is on finite-size (i.e., finite-time) effects, which are shown to exhibit different behaviors in three regimes of the size-degree plane: stationary, finite-size scaling, large deviations.

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## 1 Introduction

Complex networks have attracted much attention over the last decades. They provide a natural setting to describe many phenomena in nature and society [1, 2, 3, 4, 5]. One of the salient features of most networks, either natural and artificial, is their scalefreeness. This term refers to the broad degree distribution exhibited by these networks. The probability that a node has degree  $k$  (i.e., is connected to exactly  $k$  other nodes) is commonly observed to fall off as a power law:

$$f_k \sim k^{-\gamma}. \quad (1.1)$$

This power-law behavior, which holds in the limit of an infinitely large network, will be referred to hereafter as ‘stationary’. The exponent usually obeys  $\gamma > 2$ , so that the mean degree of the infinite network is finite. Growing networks with a preferential attachment rule, such as the well-known Barabási-Albert (BA) model [6, 7], have received a considerable interest, as they provide a natural explanation for the observed scalefreeness. The observation that preferential

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attachment generates a power-law degree distribution actually dates back to much earlier works [8,9].

Scalefree networks, being chiefly characterized by the exponent  $\gamma$  of their degree distribution, are therefore somewhat similar to equilibrium systems at their critical point. As a consequence, finite-size (i.e., finite-time) effects can be expected to yield important corrections to the asymptotic or stationary form (1.1) of the degree distribution. These effects are one of the possible causes of the cutoff phenomenon which is often observed in the degree distribution of real networks [10]. More precisely, the largest degree  $k_*(n)$  of a scalefree network at time  $n$  can be estimated by means of the following argument of extreme value statistics: it is such that the stationary probability of having  $k \geq k_*(n)$  is of order  $1/n$ . The largest degree thus grows as a power law [10,11]:

$$k_*(n) \sim n^\nu, \quad \nu = \frac{1}{\gamma - 1}. \quad (1.2)$$

This growth law is always subextensive, because one has  $\gamma > 2$ , so that  $\nu < 1$ . The cases  $2 < \gamma < 3$  (i.e.,  $1/2 < \nu < 1$ ) and  $\gamma > 3$  (i.e.,  $0 < \nu < 1/2$ ) however correspond to qualitative differences, especially in the topology and in the various dimensions of the networks [12].

The goal of this article is to provide a systematic analysis of the degree statistics of growing network models at a large but finite time  $n$ . Both the age-resolved distribution  $f_k(n, i)$  of the degree of node  $i$  at a later time  $n$  and the distribution  $f_k(n)$  of an unspecified node at time  $n$  will be considered throughout. Several works have already been devoted to this problem, both for growing networks with preferential attachment [11,13,14,15,16,17,18] and for related models of random graphs and other structures [19,20]. The present work aims at being systematic in the following three respects:

- *Models.* This work is focussed onto growing network models where a new node enters at each time step, so that nodes can be labeled by their birth date  $n$ , i.e., the time they enter the network. Node  $n$  attaches to a single earlier node ( $i = 1, \dots, n-1$ ) with probability  $p_{n,i}$ . The attachment probabilities and the initial configuration entirely define the model. The network thus obtained has the topology of a tree. The degrees  $k_i(n)$  of the nodes at time  $n$  obey the sum rule

$$\sum_{i=1}^n k_i(n) = 2L(n), \quad (1.3)$$

where  $L(n)$  is the number of links of the network at time  $n$ .

We will successively consider the following models:

- *Uniform attachment* (UA) (Section 2). The attachment probability is independent of the node, i.e., uniform over the network. This model is not scalefree. Its analysis serves as a warming up for that of the subsequent models.
- *Barabási-Albert* (BA) *model* (Section 3). The attachment probability is proportional to the degree  $k_i(n)$  of the earlier node. This well-known model [6,7] is scalefree, with exponents  $\gamma = 3$  and  $\nu = 1/2$ .

**Table 1.1** Various characteristics of the network for both initial conditions. The listed results hold irrespective of the attachment rule.

Initial condition	Case A	Case B
Topology	tree	rooted tree
Number of links at time $n$	$L^{(A)}(n) = n - 1$	$L^{(B)}(n) = n - 1/2$
Mean degree at time $n$	$\langle k^{(A)}(n) \rangle = 2 - 2/n$	$\langle k^{(B)}(n) \rangle = 2 - 1/n$
Degrees at time 1 and generating polynomials	$k_1^{(A)}(1) = 0$ $F_1^{(A)}(x) = 1$	$k_1^{(B)}(1) = 1$ $F_1^{(B)}(x) = x$
Degrees at time 2 and generating polynomials	$k_1^{(A)}(2) = k_2^{(A)}(2) = 1$ $F_2^{(A)}(x) = x$	$k_1^{(B)}(2) = 2, k_2^{(B)}(2) = 1$ $F_2^{(B)}(x) = \frac{1}{2}x(x + 1)$

– *General preferential attachment* (GPA) (Section 4). The attachment probability is proportional to the sum  $k_i(n) + c$  of the degree of the earlier node and of an additive constant  $c > -1$ . This parameter, representing the initial attractiveness of a node [11], is relevant as it yields the continuously varying exponents  $\gamma = c + 3$  and  $\nu = 1/(c + 2)$ . The BA and UA model are respectively recovered when  $c = 0$  and  $c \rightarrow \infty$ .

• *Regimes*. For each model, the following three regimes will be considered:

– *Stationary regime* ( $k \ll k_*(n)$ ). The degree distribution is essentially given by its stationary form (1.1), to be henceforth denoted by  $f_{k,\text{stat}}$ , in order to emphasize its belonging to the stationary regime.

– *Finite-size scaling regime* ( $k \sim k_*(n)$ ). In the scalefree cases, the degree distribution obeys a multiplicative finite-size scaling law of the form

$$f_k(n) \approx f_{k,\text{stat}} \Phi\left(\frac{k}{k_*(n)}\right). \quad (1.4)$$

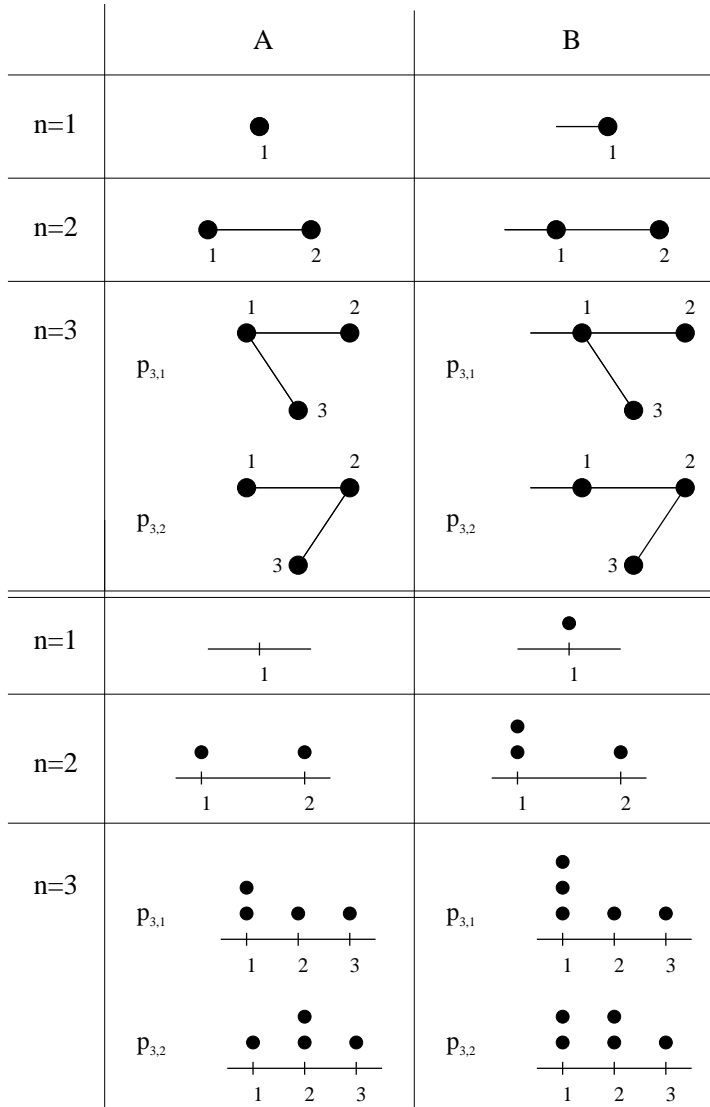
– *Large-deviation regime* ( $k_*(n) \ll k \sim n$ ). The degree distribution is usually exponentially small in  $n$ .

• *Initial conditions*. We will consider the following two initial conditions:

– *Case A*. The first node appears at time  $n = 1$  with degree  $k_1(1) = 0$ . This prescription is natural because the first node initially has no connection. All subsequent nodes appear with degree  $k_n(n) = 1$ . In particular, at time  $n = 2$  the second node connects to the first one, so that  $k_1(2) = k_2(2) = 1$ . The configuration thus obtained is the dimer configuration used e.g. in [15,16]. At time  $n$ , the network has  $L(n) = n - 1$  links. It has the topology of a tree.

– *Case B*. The first node now appears at time  $n = 1$  with degree  $k_1(1) = 1$ . This formally amounts to saying that this node is connected to a root, which does not belong to the network. It is natural to associate half a link to this fictitious connection. At time  $n = 2$  the second node connects to the first one, so that  $k_1(2) = 2$  and  $k_2(2) = 1$ . At time  $n$ , the network has  $L(n) = n - 1/2$  links. It has the topology of a rooted tree.

Table 1.1 summarizes various characteristics of the network for both initial conditions, whereas Figure 1.1 illustrates the first three steps of the network



**Fig. 1.1** First three steps of the construction of the network (upper panel) and corresponding interacting particle representation (lower panel) for both initial conditions.

construction. The upper panel shows the networks with their nodes and links. The lower panel shows the corresponding representation as an interacting particle system, where each node is viewed as a site occupied by a number of particles equal to its degree. The total number of particles in the system is therefore  $2L(n)$ . The information about the topology of the network, and especially about the genealogy of the nodes, is lost in the interacting particle

representation, but this information will not be used in the present study which is focussed on the statistics of degrees.

## 2 The uniform attachment (UA) model

The uniform attachment (UA) model is the simplest of all: the attachment probability is chosen to be uniform over all existing nodes. This section is devoted to an analytical study of the distribution of the degree of a fixed node and of an unspecified node, exactly taking into account fluctuations, finite-time effects, and the influence of the initial condition.

### 2.1 Degree statistics of a fixed node

We start with the study of the distribution of the degree  $k_i(n)$  of node  $i$  at time  $n$ . The node appearing at time  $n \geq 2$  links to any of the  $n - 1$  earlier nodes ( $i = 1, \dots, n - 1$ ) with uniform probability

$$p_{n,i} = \frac{1}{n-1}. \quad (2.1)$$

If we define the degree increment of node  $i$  at a later time  $j > i$  as

$$I_i(j) = k_i(j) - k_i(j-1) = \begin{cases} 1 & \text{with probability } p_{j,i}, \\ 0 & \text{else,} \end{cases} \quad (2.2)$$

the degree  $k_i(n)$  of node  $i$  at a later time  $n$  is given by

$$k_i(n) = k_i(i) + \sum_{j=i+1}^n I_i(j), \quad (2.3)$$

with  $k_i(i) = 1$ , except for  $i = 1$  in Case A, where  $k_1(1) = 0$  (see Table 1.1).

The mean degree  $\langle k_i(n) \rangle$  therefore reads ( $i \geq 2$ )

$$\langle k_i(n) \rangle = 1 + \sum_{j=i+1}^n \frac{1}{j-1} = H_{n-1} - H_{i-1} + 1 \approx \ln \frac{n}{i} + 1, \quad (2.4)$$

where the harmonic numbers  $H_n$  are defined in (2.20).

The distribution  $f_k(n, i) = \text{Prob}\{k_i(n) = k\}$  can be encoded in the generating polynomial

$$F_{n,i}(x) = \langle x^{k_i(n)} \rangle = \sum_{k=1}^n f_k(n, i) x^k. \quad (2.5)$$

As a consequence of (2.3), we have

$$F_{n,i}(x) = x^{k_i(i)} \prod_{j=i+1}^n \langle x^{I_i(j)} \rangle, \quad (2.6)$$

where the characteristic function of the degree increment  $I_i(j)$  assumes the simple form

$$\langle x^{I_i(j)} \rangle = 1 + (x-1)p_{j,i} = \frac{x+j-2}{j-1}, \quad (2.7)$$

irrespective of  $i$ . We thus get ( $i \geq 2$ )

$$F_{n,i}(x) = \frac{x(i-1)! \Gamma(x+n-1)}{(n-1)! \Gamma(x+i-1)}, \quad (2.8)$$

whereas only  $F_{n,1}(x)$  depends on the initial condition according to

$$F_{n,1}^{(A)}(x) = \frac{\Gamma(x+n-1)}{(n-1)! \Gamma(x)}, \quad F_{n,1}^{(B)}(x) = \frac{x \Gamma(x+n-1)}{(n-1)! \Gamma(x)}. \quad (2.9)$$

Throughout the following, the superscripts (A) and (B) mark a result which holds for a prescribed initial condition (Case A or Case B).

The product form (2.6) implies that the generating polynomials of node  $i$  at times  $n$  and  $n+1$  obey the recursion

$$F_{n+1,i}(x) = \langle x^{I_i(n+1)} \rangle F_{n,i}(x) = \frac{x+n-1}{n} F_{n,i}(x). \quad (2.10)$$

The probabilities  $f_k(n, i)$  therefore obey the recursion

$$f_k(n+1, i) = \frac{1}{n} f_{k-1}(n, i) + \left(1 - \frac{1}{n}\right) f_k(n, i), \quad (2.11)$$

with initial conditions given in Table 1.1, i.e.,

$$f_k(i, i) = \delta_{k,1} \quad (i \geq 2), \quad f_k^{(A)}(1, 1) = \delta_{k,0}, \quad f_k^{(B)}(1, 1) = \delta_{k,1}. \quad (2.12)$$

The master equations (2.11) can be directly written down by means of a simple reasoning. They provide an alternative way of describing the evolution of the degree distribution of individual nodes.

The degree distribution encoded in (2.8) has the following characteristics. The degree of node  $i$  at time  $n$  ranges from the minimal value 1 to the maximal value  $n+1-i$ . These extremal values occur with probabilities

$$f_1(n, i) = \frac{i-1}{n-1}, \quad f_{n+1-i}(n, i) = \frac{(i-1)!}{(n-1)!}. \quad (2.13)$$

The mean and the variance of the degree can be obtained by expanding the result (2.8) around  $x=1$ , using

$$\langle x^K \rangle = 1 + (x-1)\langle K \rangle + \frac{1}{2}(x-1)^2 \underbrace{\langle K^2 - K \rangle}_{\text{var } K + \langle K \rangle^2 - \langle K \rangle} + \dots, \quad (2.14)$$

where  $K$  is any random variable taking positive integer values. We thus get

$$\begin{aligned} \langle k_i(n) \rangle &= H_{n-1} - H_{i-1} + 1, \\ \text{var } k_i(n) &= H_{n-1} - H_{n-1}^{(2)} - H_{i-1} + H_{i-1}^{(2)}, \end{aligned} \quad (2.15)$$

where the harmonic numbers  $H_n$  and  $H_n^{(2)}$  are defined in (2.20). The above results hold irrespective of the initial condition. The first one coincides with (2.4).

In the scaling regime where both times  $i$  and  $n$  are large and comparable, introducing the time ratio

$$z = \frac{n}{i} \geq 1, \quad (2.16)$$

the expressions (2.15) yield

$$\langle k_i(n) \rangle \approx \ln z + 1, \quad \text{var } k_i(n) \approx \ln z. \quad (2.17)$$

In deriving the above results, we have used the asymptotic behavior of the digamma function  $\Psi(x) = \Gamma'(x)/\Gamma(x)$  and of the trigamma function  $\Psi'(x)$  as  $x \rightarrow \infty$ :

$$\Psi(x) = \ln x - \frac{1}{2x} + \dots, \quad \Psi'(x) = \frac{1}{x} + \frac{1}{2x^2} + \dots, \quad (2.18)$$

as well as their values at integers:

$$\Psi(n) = H_{n-1} - \gamma_E, \quad \Psi'(n) = \frac{\pi^2}{6} - H_{n-1}^{(2)}, \quad (2.19)$$

where

$$H_n = \sum_{i=1}^n \frac{1}{i}, \quad H_n^{(2)} = \sum_{i=1}^n \frac{1}{i^2} \quad (2.20)$$

are the harmonic numbers of the first and second kind, and  $\gamma_E$  is Euler's constant.

The entire degree distribution can be characterized in the scaling regime. Equation (2.8) indeed yields

$$F_{n,i}(x) \approx x e^{(x-1) \ln z}, \quad (2.21)$$

irrespective of the initial condition. We recognize the generating function of a Poissonian distribution with parameter  $\lambda = \ln z$ , up to a shift by one unit. We thus obtain [15,21]

$$f_k(n, i) \approx \frac{(\ln z)^{k-1}}{z (k-1)!}. \quad (2.22)$$

## 2.2 Degree statistics of the whole network

We now turn to the degree distribution of the whole network at time  $n$ ,  $f_k(n) = \text{Prob}\{k(n) = k\}$ , where  $k(n)$  stands for the degree of an unspecified node. We have

$$f_k(n) = \frac{1}{n} \sum_{i=1}^n f_k(n, i). \quad (2.23)$$

The corresponding generating polynomials,

$$F_n(x) = \langle x^{k(n)} \rangle = \sum_{k=1}^n f_k(n) x^k = \frac{1}{n} \sum_{i=1}^n F_{n,i}(x), \quad (2.24)$$

obey the recursion

$$(n+1)F_{n+1}(x) = (x+n-1)F_n(x) + x, \quad (2.25)$$

or equivalently

$$(n+1)f_k(n+1) = f_{k-1}(n) + (n-1)f_k(n) + \delta_{k,1}, \quad (2.26)$$

with initial conditions given in Table 1.1, i.e.,

$$f_k^{(A)}(1) = \delta_{k,0}, \quad f_k^{(B)}(1) = \delta_{k,1}. \quad (2.27)$$

The recursion (2.25) has a non-polynomial solution, independent of  $n$ ,

$$F_{\text{stat}}(x) = \frac{x}{2-x}, \quad (2.28)$$

describing the stationary degree distribution on an infinitely large network:

$$f_{k,\text{stat}} = \frac{1}{2^k} \quad (k \geq 1). \quad (2.29)$$

The solution of (2.25) reads

$$\begin{aligned} F_n^{(A)}(x) &= \frac{x}{2-x} + \frac{2(1-x)}{2-x} \frac{\Gamma(x+n-1)}{n!\Gamma(x)}, \\ F_n^{(B)}(x) &= \frac{x}{2-x} + \frac{x(1-x)}{2-x} \frac{\Gamma(x+n-1)}{n!\Gamma(x)}. \end{aligned} \quad (2.30)$$

The polynomials  $F_n^{(A)}(x)$  and  $F_n^{(B)}(x)$  have respective degrees  $n-1$  and  $n$ . The first of them which are not listed in Table 1.1 read

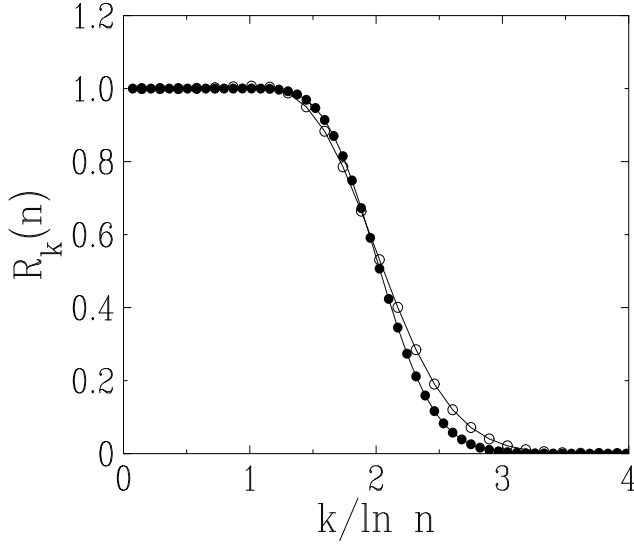
$$\begin{aligned} F_3^{(A)}(x) &= \frac{1}{3}x(x+2), & F_3^{(B)}(x) &= \frac{1}{6}x(x^2+2x+3), \\ F_4^{(A)}(x) &= \frac{1}{12}x(x^2+4x+7), & F_4^{(B)}(x) &= \frac{1}{24}x(x+3)(x^2+x+4). \end{aligned} \quad (2.31)$$

The degree  $k(n)$  at time  $n$  ranges from the minimal value 1 to the maximal value  $n-1$  (Case A) or  $n$  (Case B). These extremal values occur with the following probabilities ( $n \geq 2$ )

$$f_1^{(A)}(n) = \frac{1}{2} + \frac{1}{n(n-1)}, \quad f_1^{(B)}(n) = \frac{1}{2}, \quad f_{n-1}^{(A)}(n) = \frac{2}{n!}, \quad f_n^{(B)}(n) = \frac{1}{n!}. \quad (2.32)$$

We now turn to the finite-size scaling behavior of the degree distribution when both  $k$  and  $n$  are large. As anticipated in the Introduction, it is to be expected that the probabilities  $f_k(n)$  are close to their limits ( $f_k(n) \approx f_{k,\text{stat}}$ ) for  $n$  large at fixed degree  $k$ , and more generally in the stationary regime where  $k$  is much smaller than some characteristic crossover degree  $k_*(n)$ . Conversely, the probabilities  $f_k(n)$  are expected to be negligible ( $f_k(n) \ll f_{k,\text{stat}}$ ) for  $k$  large enough at fixed time  $n$ , and more generally in the large-deviation regime where  $k_*(n) \ll k \sim n$ . The crossover scale  $k_*(n)$  can be estimated as  $k_*(n) \approx \langle k_1(n) \rangle$  (see (2.4)). Nodes with highest degrees are indeed typically





**Fig. 2.1** Plot of the ratios  $R_k(n)$  against  $k/\ln n$  (see (2.34)), for the UA model with initial condition A, at times  $n = 10^3$  (empty symbols) and  $n = 10^6$  (full symbols).

expected to be the oldest ones. An alternative route consists in using the argument of extreme value statistics alluded to in the Introduction: the largest degree  $k_*$  at time  $n$  is such that the stationary probability of having  $k \geq k_*$  is of order  $1/n$ . Both approaches consistently yield

$$k_*(n) \sim \ln n. \quad (2.33)$$

Finite-size effects are best revealed by considering the ratios

$$R_k(n) = \frac{f_k(n)}{f_{k,\text{stat}}} = 2^k f_k(n). \quad (2.34)$$

These ratios are expected to fall off to zero for  $k$  of the order of  $k_*(n) \sim \ln n$ . Figure 2.1 shows a plot of the ratios  $R_k(n)$  against  $k/\ln n$ , for times  $n = 10^3$  and  $n = 10^6$  in Case A. Numerically exact values of the  $f_k(n)$  are obtained by iterating (2.26). A steeper and steeper crossover is clearly observed.

In order to get some quantitative information on the observed crossover, it is advantageous to introduce the differences  $d_k(n) = R_{k-1}(n) - R_k(n)$  for  $k \geq 2$ , completed by  $d_1(n) = 1 - R_1(n)$ , i.e.,  $R_0(n) = 1$ . Although the  $d_k(n)$  are not positive, most of them are, and they sum up to unity, so that it is tempting to think of them as a narrow probability distribution living in the crossover region. The generating function of the  $d_k(n)$  reads

$$D_n(x) = \sum_{k \geq 1} d_k(n) x^k = (x-1)F_n(2x) + x. \quad (2.35)$$

The above picture suggests to define the crossover scale as the first moment

$$k_\star = \mu(n) = \sum_{k \geq 1} k d_k(n) = D'_n(1), \quad (2.36)$$

and the squared width of the crossover front as the variance

$$\sigma^2(n) = \sum_{k \geq 1} k^2 d_k(n) - \mu(n)^2 = D''_n(1) + \mu(n) - \mu(n)^2. \quad (2.37)$$

Equations (2.30), (2.35) yield

$$\mu^{(A)}(n) = 2H_n \approx 2(\ln n + \gamma_E), \quad \mu^{(B)}(n) = 2H_n + 1 \approx 2(\ln n + \gamma_E) + 1, \quad (2.38)$$

and

$$\sigma^2(n) = 2H_n - 4H_n^{(2)} \approx 2(\ln n + \gamma_E - \pi^2/3), \quad (2.39)$$

the latter result being independent of the initial condition.

The crossover scale is thus  $k_\star \approx 2 \ln n$ , whereas the width of the crossover front grows as  $\sigma(n) \approx (2 \ln n)^{1/2}$ . These predictions are in agreement with the observations which can be made on Figure 2.1, namely that the crossover takes place around  $k/\ln n = 2$ , and that it becomes steeper at larger times, as its relative width falls off, albeit very slowly, as  $(\ln n)^{-1/2}$ .

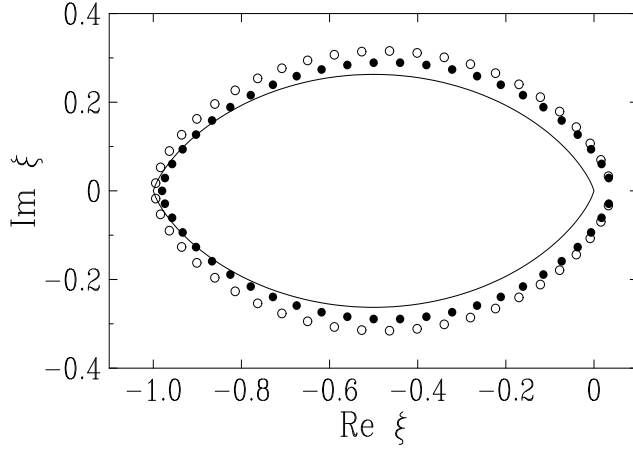
Another illustration of finite-size effects is provided by the complex zeros of the polynomials  $F_n(x)$ . The location of these zeros indeed shows how fast the degree distribution of finite networks, encoded in the polynomials  $F_n(x)$ , converges to the stationary distribution, encoded in the function  $F_{\text{stat}}(x)$ . For  $n \geq 2$ ,  $F_n^{(A)}(x)$  and  $F_n^{(B)}(x)$  have one trivial zero at  $x = 0$ , and respectively  $n - 2$  and  $n - 1$  non-trivial ones. The explicit expressions (2.30) allow one to find the asymptotic locus of the zeros as follows. The most rapidly varying part of these results is the rightmost ratio, so that the zeros are asymptotically located on the curve with equation  $|\Gamma(x + n - 1)/(n! \Gamma(x))| = 1$ . Setting

$$x = n\xi, \quad (2.40)$$

and using Stirling's formula, we can recast the above estimate as

$$\text{Re}[(1 + \xi) \ln(1 + \xi) - \xi \ln \xi] = 0. \quad (2.41)$$

The non-trivial zeros of the polynomials  $F_n(x)$  are thus predicted to escape to infinity linearly with time  $n$ . Once rescaled by  $n$  according to (2.40), they accumulate onto a well-defined limiting curve in the complex  $\xi$ -plane. This curve, with equation (2.41), has the shape of a lens connecting the points  $-1$  and  $0$ . Figure 2.2 illustrates this result with data at time  $n = 50$  for both initial conditions. The polynomials  $F_n(x)$  converge to the stationary function  $F_{\text{stat}}(x)$  whenever the complex ratio  $\xi = x/n$  lies within the lens. Otherwise they diverge exponentially with  $n$ .



**Fig. 2.2** Plot of the non-trivial zeros of the polynomials  $F_n(x)$  for the UA model, in the complex plane of the rescaled variable  $\xi = x/n$ . Symbols: zeros for  $n = 50$  in Case A (empty symbols) and Case B (full symbols). Line: limiting curve with equation (2.41).

A related issue concerns the behavior of the probability  $f_k(n)$  of having a very large degree, of order  $k \sim n$ , much larger than  $k_*(n) \sim \ln n$ . Considering Case A for definiteness, the expression (2.30) leads to the exact contour-integral representation

$$f_k^{(A)}(n) = \oint \frac{dx}{2\pi i x^{k+1}} \left( \frac{x}{2-x} + \frac{2(1-x)}{2-x} \frac{\Gamma(x+n-1)}{n! \Gamma(x)} \right). \quad (2.42)$$

The presence of gamma functions suggests to look for a saddle point  $x_s$  proportional to  $n$ . Setting  $\zeta = k/n$ , we indeed find  $x_s = n/v$ , where  $\zeta$  and  $v$  are related through

$$\zeta = \frac{\ln(v+1)}{v}. \quad (2.43)$$

We thus obtain the following large-deviation estimate

$$f_k(n) \sim \exp\left(-n(\zeta \ln n + S(\zeta))\right), \quad (2.44)$$

where the exponent has a usual contribution in  $n$  and a less usual one in  $n \ln n$ . The term linear in  $n$  involves a large-deviation function  $S(\zeta)$ , which is obtained in the following form, parametrized by  $v$ :

$$S(\zeta) = \frac{1}{v} (v \ln v - \ln v \ln(v+1) - (v+1) \ln(v+1)). \quad (2.45)$$

This function decreases from  $S(0) = 0$  to  $S(1) = -1$ . The resulting behavior at  $\zeta = 1$ , i.e.,  $\exp(-n(\ln n - 1))$ , is in agreement with the inverse factorial expressions (2.32).

### 3 Linear preferential attachment: the Barabási-Albert (BA) model

The Barabási-Albert (BA) model is the simplest of the models with preferential attachment: each new node connects to earlier nodes with a probability proportional to their degrees. The probability that node  $n$  connects to an earlier node  $i$  thus reads

$$p_{n,i} = \frac{k_i(n-1)}{Z(n-1)}, \quad (3.1)$$

where  $k_i(n-1)$  is the degree of node  $i$  at time  $n-1$ , i.e., before node  $n$  enters the network. The partition function in the denominator,

$$Z(n) = \sum_{i=1}^n k_i(n) = 2L(n) \quad (3.2)$$

(see (1.3)), ensures that the attachment probabilities add up to unity.

In the following we analyze the BA model along the lines of the previous section, keeping consistent notations as much as possible. The dependence of the attachment probability  $p_{n,i}$  on the degree  $k_i(n-1)$  however makes the problem more difficult than the previous one of a uniform attachment.

#### 3.1 Degree statistics of a fixed node

Let us again begin with the distribution  $f_k(n, i) = \text{Prob}\{k_i(n) = k\}$  of the degree of node  $i$  at time  $n$ .

A first estimate of the degree  $k_i(n)$  is provided by the following recursion relation for the mean degree  $\langle k_i(n) \rangle$ , which is a consequence of (2.2):

$$\langle k_i(n) \rangle = \langle k_i(n-1) \rangle + \langle p_{n,i} \rangle = \left(1 + \frac{1}{Z(n-1)}\right) \langle k_i(n-1) \rangle. \quad (3.3)$$

In the scaling regime where both  $i$  and  $n$  are large, using the expressions of the partition function given in Table 1.1, i.e.,

$$Z^{(A)}(n) = 2n - 2, \quad Z^{(B)}(n) = 2n - 1, \quad (3.4)$$

the above relation becomes the differential equation

$$\frac{\partial \langle k_i(n) \rangle}{\partial n} \approx \frac{\langle k_i(n) \rangle}{2n}, \quad (3.5)$$

which yields

$$\langle k_i(n) \rangle \approx \left(\frac{n}{i}\right)^{1/2}. \quad (3.6)$$

The generating polynomials  $F_{n,i}(x)$  and  $F_{n+1,i}(x)$  which encode the distribution of the degree of node  $i$  at successive times  $n$  and  $n+1$  obey the recursion formula:

$$\begin{aligned} F_{n+1,i}(x) &= \langle x^{k_i(n+1)} \rangle = \langle x^{I_i(n+1)} x^{k_i(n)} \rangle \\ &= \langle (1 + (x-1)p_{n+1,i}) x^{k_i(n)} \rangle \\ &= \left\langle \left( 1 + \frac{x-1}{Z(n)} k_i(n) \right) x^{k_i(n)} \right\rangle, \end{aligned} \quad (3.7)$$

i.e.,

$$F_{n+1,i}(x) = F_{n,i}(x) + \frac{x(x-1)}{Z(n)} \frac{dF_{n,i}(x)}{dx}, \quad (3.8)$$

where  $Z(n)$  is given by (3.4). The probabilities  $f_k(n, i)$  themselves therefore obey the recursion

$$f_k(n+1, i) = \frac{k-1}{Z(n)} f_{k-1}(n, i) + \left( 1 - \frac{k}{Z(n)} \right) f_k(n, i), \quad (3.9)$$

with initial conditions (2.12). The initial condition for Case A should be taken at time  $n=2$ , in order to avoid indeterminate expressions, as  $Z^{(A)}(1) = 0$ .

In order to solve the recursion (3.8), we perform the rational change of variable from  $x$  to  $u$  such that

$$u = \frac{x}{1-x}, \quad x = \frac{u}{u+1}, \quad x(x-1) \frac{d}{dx} = -u \frac{d}{du}. \quad (3.10)$$

Introducing the notation  $\hat{F}_{n,i}(u) = F_{n,i}(x)$ , the recursion (3.8) reads

$$\hat{F}_{n+1,i}(u) = \hat{F}_{n,i}(u) - \frac{u}{Z(n)} \frac{d\hat{F}_{n,i}(u)}{du}. \quad (3.11)$$

It is then advantageous to introduce the Mellin transform  $M_{n,i}(s)$  of  $\hat{F}_{n,i}(u)$ , defined as

$$M_{n,i}(s) = \int_0^\infty \hat{F}_{n,i}(u) u^{-s-1} du. \quad (3.12)$$

The inverse transform reads

$$\hat{F}_{n,i}(u) = \int_C \frac{ds}{2\pi i} M_{n,i}(s) u^s, \quad (3.13)$$

where  $C$  is a vertical contour in the complex  $s$ -plane whose position will be defined in a while. The virtue of the Mellin transformation is that the recursion (3.11) simplifies to

$$M_{n+1,i}(s) = \left( 1 - \frac{s}{Z(n)} \right) M_{n,i}(s), \quad (3.14)$$

with initial condition  $M_{i,i}(s) = X_0(s)$  for  $i \geq 2$ , with

$$X_0(s) = \int_0^\infty x(u) u^{-s-1} du = \int_0^1 x^{-s}(1-x)^{s-1} dx = \frac{\pi}{\sin \pi s} \quad (3.15)$$

for  $0 < \text{Re } s < 1$ . Hereafter the contour  $C$  is assumed to be in that strip. We thus get ( $i \geq 2$ )

$$\begin{aligned} M_{n,i}^{(A)}(s) &= \frac{\Gamma(n - \frac{s}{2} - 1) \Gamma(i - 1)}{\Gamma(i - \frac{s}{2} - 1) \Gamma(n - 1)} X_0(s), \\ M_{n,i}^{(B)}(s) &= \frac{\Gamma(n - \frac{s}{2} - \frac{1}{2}) \Gamma(i - \frac{1}{2})}{\Gamma(i - \frac{s}{2} - \frac{1}{2}) \Gamma(n - \frac{1}{2})} X_0(s). \end{aligned} \quad (3.16)$$

These product formulas in the Mellin variable  $s$  are reminiscent of (2.8).

The mean and the variance of the degree of node  $i$  at time  $n$  can be extracted from these results as follows. The identity (2.14) yields

$$\hat{F}_{n,i}(u) = 1 - \frac{\langle k_i(n) \rangle}{u} + \frac{\langle k_i(n)^2 \rangle + \langle k_i(n) \rangle}{2u^2} + \dots \quad (3.17)$$

as  $u \rightarrow +\infty$ . Furthermore the coefficients of  $1/u$  and  $1/u^2$  are respectively the residues of  $M_{n,i}(s)$  at  $s = -1$  and  $s = -2$ . We thus obtain

$$\langle k_i^{(A)}(n) \rangle = \frac{\Gamma(n - \frac{1}{2}) \Gamma(i - 1)}{\Gamma(i - \frac{1}{2}) \Gamma(n - 1)}, \quad \langle k_i^{(B)}(n) \rangle = \frac{\Gamma(n) \Gamma(i - \frac{1}{2})}{\Gamma(i) \Gamma(n - \frac{1}{2})} \quad (3.18)$$

and

$$\begin{aligned} \text{var } k_i^{(A)}(n) &= 2 \frac{n-1}{i-1} - \langle k_i^{(A)}(n) \rangle^2 - \langle k_i^{(A)}(n) \rangle, \\ \text{var } k_i^{(B)}(n) &= 2 \frac{2n-1}{2i-1} - \langle k_i^{(B)}(n) \rangle^2 - \langle k_i^{(B)}(n) \rangle. \end{aligned} \quad (3.19)$$

In the scaling regime where both times  $i$  and  $n$  are large and comparable, introducing the time ratio  $z = n/i$  (see (2.16)), the above results yield

$$\langle k_i(n) \rangle \approx z^{1/2}, \quad \text{var } k_i(n) \approx z^{1/2}(z^{1/2} - 1), \quad (3.20)$$

irrespective of the initial condition. The mean degree is in agreement with the estimate (3.6). The entire degree distribution can actually be derived in the scaling regime. Equation (3.16) indeed yields

$$M_{n,i}(s) \approx z^{-s/2} \frac{\pi}{\sin \pi s}. \quad (3.21)$$

We thus obtain

$$F_{n,i}(x) \approx \frac{x}{x + z^{1/2}(1-x)} \quad (3.22)$$

and finally

$$f_k(n, i) \approx z^{-1/2} (1 - z^{-1/2})^{k-1}. \quad (3.23)$$

The degree distribution is therefore found to be asymptotically geometric, irrespective of the initial condition [15, 21].

### 3.2 Degree statistics of the whole network

We now turn to the degree distribution  $f_k(n) = \text{Prob}\{k(n) = k\}$ , where  $k(n)$  stands for the degree of an unspecified node.

The generating polynomials  $F_n(x)$  obey the recursion

$$(n+1)F_{n+1}(x) = nF_n(x) + n \frac{x(x-1)}{Z(n)} \frac{dF_n(x)}{dx} + x, \quad (3.24)$$

where  $Z(n)$  is again given by (3.4), and with initial conditions given in Table 1.1. The probabilities  $f_k(n)$  themselves obey the recursion

$$(n+1)f_k(n+1) = \frac{k-1}{Z(n)} n f_{k-1}(n) + \left(1 - \frac{k}{Z(n)}\right) n f_k(n) + \delta_{k,1}. \quad (3.25)$$

The first generating polynomials which depend on the attachment rule read

$$\begin{aligned} F_3^{(A)}(x) &= \frac{1}{3} x(x+2), & F_3^{(B)}(x) &= \frac{1}{9} x(2x^2 + 2x + 5), \\ F_4^{(A)}(x) &= \frac{1}{8} x(x^2 + 2x + 5), & F_4^{(B)}(x) &= \frac{1}{60} x(6x^3 + 8x^2 + 11x + 35). \end{aligned} \quad (3.26)$$

The stationary degree distribution  $f_{k,\text{stat}}$  can be determined as the solution of (3.25) which becomes independent of  $n$  for large  $n$ . We thus get

$$(k+2)f_{k,\text{stat}} = (k-1)f_{k-1,\text{stat}} + 2\delta_{k,1}, \quad (3.27)$$

hence [11,14,15]

$$f_{k,\text{stat}} = \frac{4}{k(k+1)(k+2)}. \quad (3.28)$$

An alternative approach consists in looking for the asymptotic generating function  $F_{\text{stat}}(x)$  as the solution of (3.24) which becomes independent of  $n$  for large  $n$ . We thus obtain the differential equation

$$x(1-x)F'_{\text{stat}}(x) + 2F_{\text{stat}}(x) = 2x, \quad (3.29)$$

which has for solution

$$F_{\text{stat}}(x) = 3 - \frac{2}{x} - \frac{2(1-x)^2}{x^2} \ln(1-x). \quad (3.30)$$

Expanding this result as a power series in  $x$  allows one to recover (3.28).

The recursion (3.24) for the generating polynomials  $F_n(x)$  can be solved along the lines of the above solution of the recursion (3.8). The Mellin transforms  $M_n(s)$  of the functions  $\hat{F}_n(u) = F_n(x)$  obey the recursion

$$(n+1)M_{n+1}(s) = \left(1 - \frac{s}{Z(n)}\right) n M_n(s) + X_0(s), \quad (3.31)$$

with initial condition  $M_2^{(A)}(s) = M_1^{(B)}(s) = X_0(s)$ . Equation (3.31) has a special solution

$$M_n(s) = \frac{Z(n)X_0(s)}{(s+2)n}, \quad (3.32)$$

whereas the general solution of the homogeneous equation shares the  $n$ -dependence of the expressions (3.16). We thus obtain

$$\begin{aligned} M_n^{(A)}(s) &= \frac{2X_0(s)}{(s+2)n} \left( n-1 + (s+1) \frac{\Gamma(n - \frac{s}{2} - 1)}{\Gamma(1 - \frac{s}{2}) \Gamma(n-1)} \right), \\ M_n^{(B)}(s) &= \frac{X_0(s)}{(s+2)n} \left( 2n-1 + (s+1) \frac{\sqrt{\pi} \Gamma(n - \frac{s}{2} - \frac{1}{2})}{\Gamma(\frac{1}{2} - \frac{s}{2}) \Gamma(n - \frac{1}{2})} \right). \end{aligned} \quad (3.33)$$

The common stationary limit of both expressions,

$$M_{\text{stat}}(s) = \frac{2X_0(s)}{s+2}, \quad (3.34)$$

is proportional to the special solution (3.32). Recalling (3.15), the inverse Mellin transform of the above result,

$$\hat{F}_{\text{stat}}(u) = 1 - \frac{2}{u} + \frac{2}{u^2} \ln(u+1), \quad (3.35)$$

is equivalent to (3.30).

The results (3.33) allow one to investigate, at least in principle, every feature of the degree distribution  $f_k(n)$ . Let us take the example of the probability  $f_1(n)$  for a node to have degree one. The inverse formula (3.13) shows that this probability is equal to minus the residue of  $M_n(s)$  at  $s = 1$ . The nature of the subleading corrections to the stationary value  $f_{1,\text{stat}} = 2/3$  depends on the initial condition. For Case A we obtain ( $n \geq 2$ )

$$f_1^{(A)}(n) = \frac{2(n-1)}{3n} + \frac{4 \Gamma(n - \frac{3}{2})}{3\sqrt{\pi} n \Gamma(n-1)} = \frac{2}{3} - \frac{2}{3n} + \frac{4}{3\sqrt{\pi} n^{3/2}} + \dots \quad (3.36)$$

More generally, all the probabilities  $f_k(n)$  exhibit a singular correction in  $n^{-3/2}$ . Case B has the remarkable property that all the probabilities  $f_k(n)$  are rational functions of time  $n$ . Their expansion at large  $n$  therefore only involves integer powers of  $1/n$ . We have e.g.

$$\begin{aligned} f_1^{(B)}(n) &= \frac{2n-1}{3n} = \frac{2}{3} - \frac{1}{3n}, \\ f_2^{(B)}(n) &= \frac{n^2 - 2n + 3}{3n(2n-3)} = \frac{1}{6} - \frac{1}{12n} + \frac{3}{8n^2} + \dots \end{aligned} \quad (3.37)$$

We now turn to the finite-size scaling behavior of the degree distribution when both  $k$  and  $n$  are large. The crossover scale  $k_*(n)$  can again be estimated either using (3.6) or by the argument of extreme value statistics. Both approaches consistently yield

$$k_*(n) \sim n^{1/2}. \quad (3.38)$$



We will now show that the degree distribution obeys the multiplicative finite-size scaling law

$$f_k(n) \approx f_{k,\text{stat}} \Phi(y), \quad y = \frac{k}{n^{1/2}}, \quad (3.39)$$

where the scaling function  $\Phi(y)$  is non-universal, in the sense that it depends on the initial condition [16, 17]. The proof of the scaling behavior (3.39) and the determination of the scaling functions  $\Phi^{(A)}(y)$  and  $\Phi^{(B)}(y)$  go as follows. Let us start with Case A. The second term of the expression (3.33) for  $M_n^{(A)}(s)$  scales as a power law for large  $n$ :

$$M_{n,\text{scal}}^{(A)}(s) \approx \frac{2(s+1)X_0(s)}{(s+2)\Gamma(1-\frac{s}{2})} n^{-s/2-1}. \quad (3.40)$$

The inverse Mellin transform of the latter formula,

$$\widehat{F}_{n,\text{scal}}^{(A)}(u) \approx \frac{1}{n} \int_C \frac{ds}{2\pi i} \frac{2(s+1)X_0(s)}{(s+2)\Gamma(1-\frac{s}{2})} \left(u/n^{1/2}\right)^s, \quad (3.41)$$

describes the scaling behavior of  $\widehat{F}_n^{(A)}(u)$  in the regime where  $u$  and  $n$  are simultaneously large, with  $u/n^{1/2}$  fixed. Finally, by inserting the above scaling estimate into the contour-integral representation

$$f_k^{(A)}(n) = \oint \frac{dx}{2\pi i} \frac{F_n^{(A)}(x)}{x^{k+1}} = \oint \frac{du}{2\pi i} \frac{\widehat{F}_n^{(A)}(u)(u+1)^{k-1}}{u^{k+1}}, \quad (3.42)$$

permuting the order of integrals, opening up the  $u$ -contour and using

$$\int_C \frac{du}{2\pi i} \frac{(u+1)^{k-1}}{u^{k-s+1}} = \frac{\Gamma(k)}{\Gamma(s)\Gamma(k-s+1)}, \quad (3.43)$$

we obtain after some algebra the scaling form (3.39), with

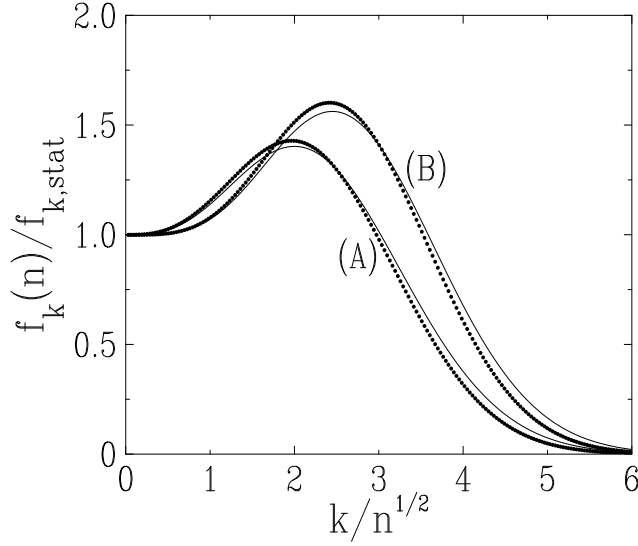
$$\Phi^{(A)}(y) = 1 + \frac{2}{\sqrt{\pi}} \int_C \frac{ds}{2\pi i} \frac{s+1}{s+2} \Gamma\left(\frac{1-s}{2}\right) \left(\frac{y}{2}\right)^{s+2}. \quad (3.44)$$

Case B can be dealt with along the same lines. We thus get the similar expression

$$\Phi^{(B)}(y) = 1 + \int_C \frac{ds}{2\pi i} \frac{s+1}{s+2} \Gamma\left(1-\frac{s}{2}\right) \left(\frac{y}{2}\right)^{s+2}. \quad (3.45)$$

The above expressions can be evaluated by closing the contour to the right and summing the residues at the poles of the gamma functions. We thus get

$$\begin{aligned} \Phi^{(A)}(y) &= 1 + \frac{8}{\sqrt{\pi}} \sum_{m \geq 0} \frac{(-1)^m (m+1)}{(2m+3)m!} \left(\frac{y}{2}\right)^{2m+3}, \\ \Phi^{(B)}(y) &= 1 + \sum_{m \geq 0} \frac{(-1)^m (m+1)(2m+3)}{(m+2)!} \left(\frac{y}{2}\right)^{2m+4}, \end{aligned} \quad (3.46)$$



**Fig. 3.1** Plot of the ratios  $f_k(n)/f_{k,\text{stat}}$  against the scaling variable  $y = k/n^{1/2}$ , for the BA model at time  $n = 10^3$  (symbols) for both initial conditions. Full lines: asymptotic scaling functions  $\Phi^{(A)}(y)$  and  $\Phi^{(B)}(y)$ .

i.e., finally

$$\begin{aligned}\Phi^{(A)}(y) &= \text{erfc}\left(\frac{y}{2}\right) + \frac{y}{\sqrt{\pi}} \left(1 + \frac{y^2}{2}\right) e^{-y^2/4}, \\ \Phi^{(B)}(y) &= \left(1 + \frac{y^2}{4} + \frac{y^4}{8}\right) e^{-y^2/4},\end{aligned}\tag{3.47}$$

where  $\text{erfc}$  denotes the complementary error function. The above expression for  $\Phi^{(A)}$  can be found in [16,17], whereas that for  $\Phi^{(B)}$  has been shown in [13] to hold for a slightly different attachment rule and initial condition.

Figure 3.1 shows a plot of the ratios  $f_k(n)/f_{k,\text{stat}}$ , against the scaling variable  $y = k/n^{1/2}$ , at time  $n = 10^3$  for both initial conditions. Exact values for the  $f_k(n)$  are obtained by iterating (3.25). The data are well described by the predicted finite-size scaling functions  $\Phi^{(A)}(y)$  and  $\Phi^{(B)}(y)$ , shown as full lines.

Both scaling functions share similar qualitative features. They start from the value 1 at  $y = 0$ , increase to a maximum, which is reached for  $y = 2$  in Case A and for  $y = \sqrt{6}$  in Case B, and fall off as  $\exp(-y^2/4)$ . They however differ at the quantitative level, both at small and large values of  $y$ :

$$\begin{aligned}\Phi^{(A)}(y) &= 1 + \frac{y^3}{3\sqrt{\pi}} + \dots, & \Phi^{(B)}(y) &= 1 + \frac{3y^4}{32} + \dots, \\ \Phi^{(A)}(y) &\approx \frac{y^3}{2\sqrt{\pi}} e^{-y^2/4}, & \Phi^{(B)}(y) &\approx \frac{y^4}{8} e^{-y^2/4}.\end{aligned}\tag{3.48}$$

Apart from the additive constant 1, the scaling functions  $\Phi^{(A)}$  and  $\Phi^{(B)}$  are respectively an odd and an even function of  $y$ . This is the transcription in the

finite-size scaling regime of the phenomenon underlined when discussing (3.36) and (3.37). In particular, the first correction term at small  $y$  is in  $y^3$  for  $\Phi^{(A)}$ , and in  $y^4$  for  $\Phi^{(B)}$ .

Let us again close up with the location of the complex zeros of the polynomials  $F_n(x)$ . Considering Case A for definiteness, the result (3.33) can be recast as the exact formula

$$\hat{F}_n^{(A)}(u) - \frac{n-1}{n} \hat{F}_{\text{stat}}(u) = \frac{1}{n} \int_C \frac{ds}{1} \frac{s+1}{s+2} \frac{\Gamma(n - \frac{s}{2} - 1)}{\Gamma(1 - \frac{s}{2}) \Gamma(n-1)} \frac{u^s}{\sin \pi s}. \quad (3.49)$$

The growth of this expression with  $n$  for a fixed value of the complex variable  $u$  can be investigated by means of the saddle-point approximation. The presence of gamma functions again suggest to look for a saddle point  $s_s$  proportional to  $n$ . Skipping details, let us mention that we find  $s_s \approx 2n/(1-u^2)$ , so that the right-hand side of (3.49) can be estimated as

$$\hat{F}_{n,\text{sing}}(u) \sim \left(1 - \frac{1}{u^2}\right)^{-n}, \quad (3.50)$$

with exponential accuracy. The asymptotic locus of the complex zeros is then naturally given by the condition that the above estimate neither falls off nor grows exponentially. We thus obtain  $|1 - 1/u^2| = 1$ . The relevant part of this locus can be parametrized by an angle  $0 \leq \theta \leq 2\pi$  as

$$u = (1 - e^{-i\theta})^{-1/2}, \quad x = \frac{1}{1 - (1 - e^{-i\theta})^{1/2}}. \quad (3.51)$$

This closed curve in the  $x$ -plane has a cusp at the point  $x = 1$ , corresponding to the scaling regime, with a right opening angle. We have indeed  $x - 1 \approx (e^{i\pi/2}\theta)^{1/2}$  as  $\theta \rightarrow 0$ . Figure 3.2 illustrates this result with data at time  $n = 50$  for both initial conditions. The polynomials  $F_n(x)$  converge to the stationary series  $F_{\text{stat}}(x)$  whenever the complex variable  $x$  lies within the closed curve shown on the figure. Otherwise they diverge exponentially with  $n$ .

The exponential estimate (3.50) has another virtue. By inserting it into the contour-integral representation (3.42), we obtain

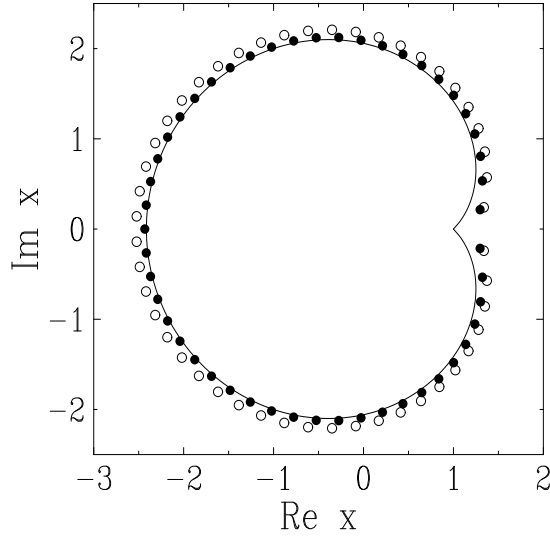
$$f_k(n) \sim \oint \frac{du}{2\pi i} \left(\frac{u+1}{u}\right)^k \left(1 - \frac{1}{u^2}\right)^{-n}. \quad (3.52)$$

This integral can in turn be investigated by means of the saddle-point approximation. The result is the following large-deviation estimate

$$f_k(n) \sim \exp(-n S(\zeta)), \quad (3.53)$$

where  $\zeta = k/n$ , and where the large-deviation function  $S(\zeta)$  reads

$$S(\zeta) = (1 - \zeta) \ln(1 - \zeta) - (2 - \zeta) \ln \frac{2 - \zeta}{2}. \quad (3.54)$$



**Fig. 3.2** Plot of the non-trivial zeros of the polynomials  $F_n(x)$  for the BA model in the complex  $x$ -plane. Symbols: zeros for  $n = 50$  in Case A (empty symbols) and Case B (full symbols). Line: limiting curve with equation (3.51).

The formula (3.53) describes, with exponential accuracy, the degree distribution in the whole large-deviation regime where  $k$  and  $n$  are comparable. The quadratic growth  $S(\zeta) \approx \zeta^2/4$  at small  $\zeta$  matches the fall-off of the finite-size scaling functions  $\Phi^{(A)}(y) \sim \Phi^{(B)}(y) \sim \exp(-y^2/4)$  (see (3.48)). The maximal value  $S(1) = \ln 2$  describes the fall-off  $f_k(n) \sim 2^{-n}$  of the probability of having a degree  $k$  equal to its maximal value ( $k = n$  or  $k = n - 1$ ).

#### 4 The general preferential attachment (GPA) model

We now consider the general preferential attachment (GPA) rule, where the attachment probability to a node is proportional to the sum  $k_i(n) + c$  of the degree of the earlier node and of an additive constant  $c$ , representing the initial attractiveness of the node [11]. This attachment rule interpolates between the uniform attachment rule, which is recovered in the  $c \rightarrow \infty$  limit, and the BA model, which corresponds to  $c = 0$ . It can actually be continued on the other side of the BA model, as  $c$  can be chosen in the range  $-1 < c < \infty$ . The GPA model thus defined is scalefree for any finite value of  $c$ , with the continuously varying exponents  $\gamma = c + 3$  and  $\nu = 1/(c + 2)$ .

The probability that node  $n$  connects to an earlier node  $i$  thus reads

$$p_{n,i} = \frac{k_i(n-1) + c}{Z(n-1)}, \quad (4.1)$$

where  $k_i(n-1)$  is the degree of node  $i$  at time  $n-1$ , and the partition function in the denominator,

$$Z(n) = \sum_{i=1}^n (k_i(n) + c) = 2L(n) + cn \quad (4.2)$$

(see (1.3)), ensures that the attachment probabilities add up to unity.

In the following we analyze the GPA model along the very lines of the previous section.

#### 4.1 Degree statistics of a fixed node

Let us again begin with the distribution  $f_k(n, i) = \text{Prob}\{k_i(n) = k\}$  of the degree of node  $i$  at time  $n$ .

A first estimate of the degree  $k_i(n)$  is provided by the product formula (3.3) for the mean degree  $\langle k_i(n) \rangle$ , which still holds in the present case. In the scaling regime where both  $i$  and  $n$  are large, the latter relation becomes the differential equation

$$\frac{\partial \langle k_i(n) \rangle}{\partial n} \approx \frac{\langle k_i(n) \rangle + c}{(c+2)n}, \quad (4.3)$$

which yields

$$\langle k_i(n) \rangle \approx (c+1) \left( \frac{n}{i} \right)^{1/(c+2)} - c. \quad (4.4)$$

As anticipated, this expressions exhibits a power-law growth with exponent  $\nu = 1/(c+2)$  in the range  $0 < \nu < 1$ .

The generating polynomials  $F_{n,i}(x)$  and  $F_{n+1,i}(x)$  associated with the degree of node  $i$  at successive times  $n$  and  $n+1$  obey the recursion formula:

$$\begin{aligned} F_{n+1,i}(x) &= \langle x^{k_i(n+1)} \rangle = \langle x^{I_i(n+1)} x^{k_i(n)} \rangle \\ &= \langle (1 + (x-1)p_{n+1,i}) x^{k_i(n)} \rangle \\ &= \left\langle \left( 1 + \frac{x-1}{Z(n)} (k_i(n) + c) \right) x^{k_i(n)} \right\rangle, \end{aligned} \quad (4.5)$$

i.e.,

$$F_{n+1,i}(x) = F_{n,i}(x) + \frac{x-1}{Z(n)} \left( c F_{n,i}(x) + x \frac{dF_{n,i}(x)}{dx} \right), \quad (4.6)$$

where

$$Z^{(A)}(n) = (c+2)n - 2, \quad Z^{(B)}(n) = (c+2)n - 1. \quad (4.7)$$

The probabilities  $f_k(n, i)$  themselves therefore obey the recursion

$$f_k(n+1, i) = \frac{k+c-1}{Z(n)} f_{k-1}(n, i) + \left( 1 - \frac{k+c}{Z(n)} \right) f_k(n, i), \quad (4.8)$$

with initial conditions (2.12).

In order to solve the recursion (4.6), we again perform the change of variable (3.10) from  $x$  to  $u$ , and set

$$F_{n,i}(x) = (1-x)^{-c} \widehat{F}_{n,i}(u). \quad (4.9)$$

The recursion (4.6) then reads

$$\widehat{F}_{n+1,i}(u) = \widehat{F}_{n,i}(u) - \frac{1}{Z(n)} \left( c \widehat{F}_{n,i}(u) + u \frac{d\widehat{F}_{n,i}(u)}{du} \right). \quad (4.10)$$

We then again introduce the Mellin transform  $M_{n,i}(s)$  of  $\widehat{F}_{n,i}(u)$ , so that the recursion (4.10) simplifies to

$$M_{n+1,i}(s) = \left( 1 - \frac{s+c}{Z(n)} \right) M_{n,i}(s), \quad (4.11)$$

with initial condition  $M_{i,i}(s) = X_c(s)$  for  $i \geq 2$ , with

$$X_c(s) = \int_0^1 x^{-s} (1-x)^{s+c-1} dx = \frac{\Gamma(1-s)\Gamma(s+c)}{\Gamma(c+1)} \quad (4.12)$$

for  $-c < \operatorname{Re} s < 1$ . Hereafter the contour  $C$  is assumed to be in that strip. We thus get ( $i \geq 2$ )

$$\begin{aligned} M_{n,i}^{(A)}(s) &= \frac{\Gamma\left(n - \frac{s+c+2}{c+2}\right) \Gamma\left(i - \frac{2}{c+2}\right)}{\Gamma\left(i - \frac{s+c+2}{c+2}\right) \Gamma\left(n - \frac{2}{c+2}\right)} X_c(s), \\ M_{n,i}^{(B)}(s) &= \frac{\Gamma\left(n - \frac{s+c+1}{c+2}\right) \Gamma\left(i - \frac{1}{c+2}\right)}{\Gamma\left(i - \frac{s+c+1}{c+2}\right) \Gamma\left(n - \frac{1}{c+2}\right)} X_c(s), \end{aligned} \quad (4.13)$$

These product formulas are a generalization of (3.16). The mean and the variance of the degree of node  $i$  at time  $n$  can be extracted from these results as follows. The identity (2.14) now yields

$$\widehat{F}_{n,i}(u) = \frac{1}{u^c} - \frac{\langle k_i(n) \rangle + c}{u^{c+1}} + \frac{\langle k_i(n)^2 \rangle + (2c+1)\langle k_i(n) \rangle + c(c+1)}{2u^{c+2}} + \dots \quad (4.14)$$

as  $u \rightarrow +\infty$ . Furthermore the coefficients of this expansion are respectively the residues of  $M_{n,i}(s)$  at  $s = -c$ ,  $s = -c-1$  and  $s = -c-2$ . We thus obtain

$$\begin{aligned} \langle k_i^{(A)}(n) \rangle &= (c+1) \frac{\Gamma\left(n - \frac{1}{c+2}\right) \Gamma\left(i - \frac{2}{c+2}\right)}{\Gamma\left(i - \frac{1}{c+2}\right) \Gamma\left(n - \frac{2}{c+2}\right)} - c, \\ \langle k_i^{(B)}(n) \rangle &= (c+1) \frac{\Gamma(n) \Gamma\left(i - \frac{1}{c+2}\right)}{\Gamma(i) \Gamma\left(n - \frac{1}{c+2}\right)} - c \end{aligned} \quad (4.15)$$

and

$$\begin{aligned}
\text{var } k_i^{(A)}(n) &= (c+1)(c+2) \frac{\Gamma(n) \Gamma\left(i - \frac{2}{c+2}\right)}{\Gamma(i) \Gamma\left(n - \frac{2}{c+2}\right)} \\
&\quad - \langle k_i^{(A)}(n) \rangle^2 - (2c+1) \langle k_i^{(A)}(n) \rangle - c(c+1), \\
\text{var } k_i^{(B)}(n) &= (c+1)(c+2) \frac{\Gamma\left(n + \frac{1}{c+2}\right) \Gamma\left(i - \frac{1}{c+2}\right)}{\Gamma\left(i + \frac{1}{c+2}\right) \Gamma\left(n - \frac{1}{c+2}\right)} \\
&\quad - \langle k_i^{(B)}(n) \rangle^2 - (2c+1) \langle k_i^{(B)}(n) \rangle - c(c+1).
\end{aligned} \tag{4.16}$$

In the scaling regime where both times  $i$  and  $n$  are large and comparable, introducing the time ratio  $z = n/i$  (see (2.16)), the above results yield

$$\langle k_i(n) \rangle \approx (c+1)z^{1/(c+2)} - c, \quad \text{var } k_i(n) \approx (c+1)z^{1/(c+2)}(z^{1/(c+2)} - 1), \tag{4.17}$$

irrespective of the initial condition. The mean degree is in agreement with the estimate (4.4). The entire degree distribution can actually be derived in the scaling regime. Equation (4.13) indeed yields

$$M_{n,i}(s) \approx z^{-(s+c)/(c+2)} X_c(s). \tag{4.18}$$

We thus obtain after some algebra

$$F_{n,i}(x) \approx \frac{x}{(x + z^{1/(c+2)}(1-x))^{c+1}} \tag{4.19}$$

and finally

$$f_k(n, i) \approx z^{-(c+1)/(c+2)} (1 - z^{-1/(c+2)})^{k-1} \frac{\Gamma(k+c)}{\Gamma(k)\Gamma(c+1)}. \tag{4.20}$$

This result allows one to recover both the Poissonian law (2.22) in the  $c \rightarrow \infty$  limit and the geometric one (3.23) as  $c = 0$ .

## 4.2 Degree statistics of the whole network

We now turn to the degree distribution of the whole network at time  $n$ ,  $f_k(n) = \text{Prob}\{k(n) = k\}$ , where  $k(n)$  stands for the degree of an unspecified node.

The generating polynomials  $F_n(x)$  obey the recursion

$$(n+1)F_{n+1}(x) = nF_n(x) + \frac{n(x-1)}{Z(n)} \left( cF_n(x) + x \frac{dF_n(x)}{dx} \right) + x, \tag{4.21}$$

where  $Z(n)$  is given by (4.7), and with initial conditions given in Table 1.1. The probabilities  $f_k(n)$  themselves obey the recursion

$$(n+1)f_k(n+1) = \frac{k+c-1}{Z(n)} n f_{k-1}(n) + \left( 1 - \frac{k+c}{Z(n)} \right) n f_k(n) + \delta_{k,1}. \tag{4.22}$$

The first generating polynomials which depend on the attachment rule read

$$\begin{aligned}
F_3^{(A)}(x) &= \frac{1}{3} x(x+2), \\
F_3^{(B)}(x) &= \frac{1}{3(2c+3)} x((c+2)x^2 + 2(c+1)x + 3c+5), \\
F_4^{(A)}(x) &= \frac{1}{4(3c+4)} x((c+2)x^2 + 4(c+1)x + 7c+10), \\
F_4^{(B)}(x) &= \frac{1}{4(2c+3)(3c+5)} x((c+2)(c+3)x^3 + 4(c+1)(c+2)x^2 \\
&\quad + (c+1)(7c+11)x + (3c+5)(4c+7)).
\end{aligned} \tag{4.23}$$

The stationary degree distribution  $f_{k,\text{stat}}$  can be determined as the solution of (4.22) which becomes independent of  $n$  for large  $n$ . We thus get

$$(k+2c+2)f_{k,\text{stat}} = (k+c-1)f_{k-1,\text{stat}} + (c+2)\delta_{k,1}, \tag{4.24}$$

hence [11, 14]

$$f_{k,\text{stat}} = \frac{(c+2)\Gamma(2c+3)\Gamma(k+c)}{\Gamma(c+1)\Gamma(k+2c+3)}. \tag{4.25}$$

This result has a power-law decay at large  $k$ :

$$f_{k,\text{stat}} \approx \frac{(c+2)\Gamma(2c+3)}{\Gamma(c+1)} k^{-(c+3)}. \tag{4.26}$$

An alternative approach consists in looking for the generating function  $F_{\text{stat}}(x)$  as the stationary solution of (4.21). We thus obtain the differential equation

$$x(1-x)F'_{\text{stat}}(x) + (2c+2-cx)F_{\text{stat}}(x) = (c+2)x, \tag{4.27}$$

which is equivalent to (4.24). The solution

$$F_{\text{stat}}(x) = \frac{(c+2)(1-x)^{c+2}}{x^{2c+2}} \int_0^x \frac{y^{2c+2}}{(1-y)^{c+3}} dy \tag{4.28}$$

can be recast in terms of a hypergeometric function, which boils down to elementary functions whenever  $2c$  is an integer.

Throughout the regime where the degree  $k$  and the parameter  $c$  are both large and comparable, the expression (4.25) assumes a stationary large-deviation form,

$$f_{k,\text{stat}} \sim \exp(-c\phi(\kappa)), \tag{4.29}$$

where  $\kappa = k/c$ , and with

$$\phi(\kappa) = (\kappa+2)\ln(\kappa+2) - (\kappa+1)\ln(\kappa+1) - 2\ln 2. \tag{4.30}$$

The linear behavior  $\phi(\kappa) \approx \kappa \ln 2$  as  $\kappa \rightarrow 0$  matches the exponential decay (2.29) of the stationary distribution in the UA model, formally corresponding to  $c \rightarrow \infty$ , whereas the logarithmic growth  $\phi(\kappa) \approx \ln \kappa + 1 - 2\ln 2$  as  $\kappa \rightarrow \infty$  matches the power-law decay (4.26).



The moments of the stationary distribution,

$$m_p = \sum_{k \geq 1} k^p f_{k,\text{stat}}, \quad (4.31)$$

can be derived from (4.24), which yields the recursion

$$(c+2-p)m_p = c+2 + pcm_{p-1} + \sum_{q=0}^{p-2} \binom{p}{q} (m_{q+1} + cm_q). \quad (4.32)$$

We thus get

$$\begin{aligned} m_0 &= 1, \quad m_1 = 2, \quad m_2 = \frac{2(3c+2)}{c}, \\ m_3 &= \frac{2(13c^2+17c+6)}{c(c-1)}, \quad m_4 = \frac{2(3c+2)(25c^2+33c+14)}{c(c-1)(c-2)}, \end{aligned} \quad (4.33)$$

and so on. The power-law decay (4.26) implies that the moment  $m_p$  is convergent for  $c > p-2$ .

The recursion (4.21) for the generating polynomials  $F_n(x)$  can again be exactly solved for a finite time  $n$ . The Mellin transforms  $M_n(s)$  of the functions  $\hat{F}_n(u) = (1-x)^c F_n(x)$  obey

$$(n+1)M_{n+1}(s) = \left(1 - \frac{s+c}{Z(n)}\right) nM_n(s) + X_c(s), \quad (4.34)$$

with initial condition  $M_2^{(A)}(s) = M_1^{(B)}(s) = X_c(s)$ . Equation (4.34) has a special solution

$$M_n(s) = \frac{Z(n)X_c(s)}{(s+2c+2)n}, \quad (4.35)$$

whereas the general solution of the homogeneous equation shares the  $n$ -dependence of the expressions (4.13). We thus get

$$\begin{aligned} M_n^{(A)}(s) &= \frac{X_c(s)}{(s+2c+2)n} \\ &\times \left( (c+2)n-2+2(s+c+1) \frac{\Gamma\left(n-\frac{s+c+2}{c+2}\right) \Gamma\left(\frac{2c+2}{c+2}\right)}{\Gamma\left(1-\frac{s}{c+2}\right) \Gamma\left(n-\frac{2}{c+2}\right)} \right), \\ M_n^{(B)}(s) &= \frac{X_c(s)}{(s+2c+2)n} \\ &\times \left( (c+2)n-1+(s+c+1) \frac{\Gamma\left(n-\frac{s+c+1}{c+2}\right) \Gamma\left(\frac{c+1}{c+2}\right)}{\Gamma\left(\frac{1-s}{c+2}\right) \Gamma\left(n-\frac{1}{c+2}\right)} \right). \end{aligned} \quad (4.36)$$

In order to illustrate these general results, let us again consider the probability  $f_1(n)$  for a node to have degree one. This probability is minus the residue of  $M_n(s)$  at  $s = 1$ . For Case A we obtain ( $n \geq 2$ )

$$\begin{aligned} f_1^{(A)}(n) &= \frac{1}{(2c+3)n} \left( (c+2)n - 2 + 2(c+2) \frac{\Gamma\left(n - \frac{c+3}{c+2}\right) \Gamma\left(\frac{2c+2}{c+2}\right)}{\Gamma\left(\frac{c+1}{c+2}\right) \Gamma\left(n - \frac{2}{c+2}\right)} \right) \\ &= \frac{1}{2c+3} \left( c+2 - \frac{2}{n} + \frac{2(c+2) \Gamma\left(\frac{2c+2}{c+2}\right)}{\Gamma\left(\frac{c+1}{c+2}\right)} n^{-\frac{2c+3}{c+2}} + \dots \right), \end{aligned} \quad (4.37)$$

whereas for Case B we obtain ( $n \geq 2$ )

$$f_1^{(B)}(n) = \frac{(c+2)n - 1}{(2c+3)n}. \quad (4.38)$$

This rational expression for  $f_1^{(B)}(n)$  is however an exception. The probabilities  $f_k(n)$  indeed generically have a singular correction in  $n^{-(2c+3)/(c+2)}$  for both initial conditions, whereas only  $f_1^{(B)}(n)$  and  $f_2^{(B)}(n)$  are rational functions of time  $n$ .

We now turn to the finite-size scaling behavior of the degree distribution when both  $k$  and  $n$  are large. The crossover scale  $k_*(n)$  can again be estimated either using (4.4) or by the argument of extreme value statistics. Both approaches consistently yield

$$k_*(n) \sim n^{1/(c+2)}. \quad (4.39)$$

The degree distribution obeys a finite-size scaling law of the form

$$f_k(n) \approx f_{k,\text{stat}} \Phi(y), \quad y = \frac{k}{n^{1/(c+2)}}, \quad (4.40)$$

where the scaling function  $\Phi(y)$  again depends on the initial condition [16, 17]. The determination of the scaling functions  $\Phi^{(A)}(y)$  and  $\Phi^{(B)}(y)$  closely follows the steps of Section 3.2. We thus obtain

$$\begin{aligned} \Phi^{(A)}(y) &= 1 + \frac{2 \Gamma\left(\frac{2c+2}{c+2}\right)}{(c+2)\Gamma(2c+3)} \int_C \frac{ds}{2\pi i} \frac{s+c+1}{s+2c+2} \frac{\Gamma(1-s)}{\Gamma\left(1 - \frac{s}{c+2}\right)} y^{s+2c+2}, \\ \Phi^{(B)}(y) &= 1 + \frac{\Gamma\left(\frac{c+1}{c+2}\right)}{(c+2)\Gamma(2c+3)} \int_C \frac{ds}{2\pi i} \frac{s+c+1}{s+2c+2} \frac{\Gamma(1-s)}{\Gamma\left(\frac{1-s}{c+2}\right)} y^{s+2c+2}. \end{aligned} \quad (4.41)$$

By closing the contours to the right, we can derive the following convergent series:

$$\begin{aligned}\Phi^{(A)}(y) &= 1 + \frac{2\Gamma\left(\frac{2c+2}{c+2}\right)}{(c+2)\Gamma(2c+3)} y^{2c+3} \sum_{m \geq 0} \frac{(m+c+2)(-y)^m}{(m+2c+3)m! \Gamma\left(1 - \frac{m+1}{c+2}\right)}, \\ \Phi^{(B)}(y) &= 1 + \frac{\Gamma\left(\frac{c+1}{c+2}\right)}{(c+2)\Gamma(2c+3)} y^{2c+3} \sum_{m \geq 1} \frac{(m+c+2)(-y)^m}{(m+2c+3)m! \Gamma\left(-\frac{m}{c+2}\right)}.\end{aligned}\tag{4.42}$$

The above expression for  $\Phi^{(A)}$  can be found in [17], albeit not in a fully explicit form. It is also worth mentioning that the finite-size scaling function derived in [20] for asymmetric growing networks is different from the above one for generic values of the exponent  $\nu = 1/(c+2)$ , although it coincides for  $\nu = 1/2$  with our result (3.47) for  $\Phi^{(A)}$ .

The expressions (4.42) suggest that the derivatives  $\Phi^{(A)'}(y)$  and  $\Phi^{(B)'}(y)$  are somewhat simpler than the functions themselves. The factor  $(m+2c+3)$  is indeed chased away from the denominators under differentiation. The resulting series can be resummed by means of the identities

$$\begin{aligned}\sum_{m \geq 0} \frac{(-y)^m}{m! \Gamma\left(1 - \frac{m+1}{c+2}\right)} &= (c+2) \int_{\mathbb{C}} \frac{dz}{2\pi i} e^{-yz+z^{c+2}}, \\ \sum_{m \geq 1} \frac{(-y)^m}{m! \Gamma\left(-\frac{m}{c+2}\right)} &= y \int_{\mathbb{C}} \frac{dz}{2\pi i} e^{-yz+z^{c+2}},\end{aligned}\tag{4.43}$$

which are known e.g. in the theory of Lévy stable laws. We are thus left with the following alternative contour-integral expressions for the derivatives:

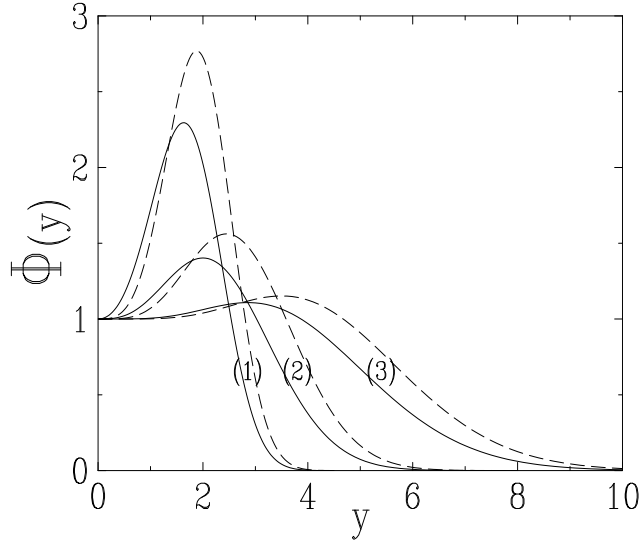
$$\begin{aligned}\Phi^{(A)'}(y) &= \frac{2\Gamma\left(\frac{2c+2}{c+2}\right)}{\Gamma(2c+3)} y^{2c+2} \int_{\mathbb{C}} \frac{dz}{2\pi i} (c+2-yz) e^{-yz+z^{c+2}}, \\ \Phi^{(B)'}(y) &= \frac{\Gamma\left(\frac{c+1}{c+2}\right)}{(c+2)\Gamma(2c+3)} y^{2c+3} \int_{\mathbb{C}} \frac{dz}{2\pi i} (c+3-yz) e^{-yz+z^{c+2}}.\end{aligned}\tag{4.44}$$

Both scaling functions start increasing from the value 1 according to the power laws

$$\begin{aligned}\Phi^{(A)}(y) &= 1 + \frac{2\Gamma\left(\frac{2c+2}{c+2}\right)}{\Gamma(2c+4)\Gamma\left(\frac{c+1}{c+2}\right)} y^{2c+3} + \dots, \\ \Phi^{(B)}(y) &= 1 + \frac{(c+3)}{2(c+2)^3\Gamma(2c+3)} y^{2c+4} + \dots,\end{aligned}\tag{4.45}$$

go through a maximum, and fall off superexponentially as

$$\Phi^{(A)}(y) \approx 2(c+2)C\Gamma\left(\frac{2c+2}{c+2}\right)\Psi(y), \quad \Phi^{(B)}(y) \approx C\Gamma\left(\frac{c+1}{c+2}\right)y\Psi(y),\tag{4.46}$$



**Fig. 4.1** Plot of the scaling functions  $\Phi^{(A)}(y)$  (full lines) and  $\Phi^{(B)}(y)$  (dashed lines) against  $y$ , for (1)  $c = -1/2$ , i.e.,  $\nu = 2/3$ ; (2)  $c = 0$ , i.e.,  $\nu = 1/2$  (the BA model); and (3)  $c = 1$ , i.e.,  $\nu = 1/3$ .

with

$$\Psi(y) = y^{2c+3-\frac{c}{2(c+1)}} \exp\left(- (c+1) \left(\frac{y}{c+2}\right)^{\frac{c+2}{c+1}}\right) \quad (4.47)$$

and

$$C = \left[ (2\pi(c+1))^{1/2} (c+2)^{\frac{2c+3}{2(c+1)}} \Gamma(2c+3) \right]^{-1}. \quad (4.48)$$

Figure 4.1 shows a plot of the scaling functions  $\Phi^{(A)}(y)$  and  $\Phi^{(B)}(y)$  for (1)  $c = -1/2$ , i.e.,  $\nu = 2/3$ ; (2)  $c = 0$ , i.e.,  $\nu = 1/2$  (the BA model); and (3)  $c = 1$ , i.e.,  $\nu = 1/3$ . The figure demonstrates that the scaling functions present a high and narrow maximum for the smaller values of  $c$ , and a direct crossover from 1 to 0 for the larger values of  $c$ . These observations can be made quantitative by means of the pseudo-moments

$$\mu_p = - \int_0^\infty \Phi'(y) y^p dy = p \int_0^\infty \Phi(y) y^{p-1} dy. \quad (4.49)$$

The integral formulas (4.44) allow one to evaluate these quantities explicitly:

$$\begin{aligned} \mu_p^{(A)} &= \frac{(p+c+1) \Gamma\left(\frac{3c+4}{c+2}\right) \Gamma(p+2c+3)}{(c+1) \Gamma\left(\frac{p+3c+4}{c+2}\right) \Gamma(2c+3)}, \\ \mu_p^{(B)} &= \frac{\Gamma\left(\frac{c+1}{c+2}\right) \Gamma(p+2c+3)}{\Gamma\left(\frac{p+c+1}{c+2}\right) \Gamma(2c+3)}. \end{aligned} \quad (4.50)$$

• For large values of  $c$  (i.e.,  $c \rightarrow \infty$ ), the model is close to the UA model. The analysis of the scaling functions will follow that of the ratios  $R_k(n)$  in the UA model, performed in Section 2.2. The crossover value of  $y$ , at which the functions exhibit a relatively sharp crossover from 1 to 0, can be estimated as  $\mu_1$ , i.e.,

$$\mu_1^{(A)} = 2c + 2\gamma_E + 2 + \dots, \quad \mu_1^{(B)} = 2c + 2\gamma_E + 3 + \dots, \quad (4.51)$$

which grows as  $2c$ , irrespective of the initial condition. Similarly, the squared width of the crossover region can be estimated as the pseudo-variance  $\sigma^2 = \mu_2 - \mu_1^2$ , i.e.,

$$\sigma^{2(A)} = 2c + 4\gamma_E + 2 - 2\pi^2/3 + \dots, \quad \sigma^{2(B)} = 2c + 4\gamma_E + 3 - 2\pi^2/3 + \dots, \quad (4.52)$$

which also grows as  $2c$ , irrespective of the initial condition.

• For small values of  $c$  (i.e.,  $c \rightarrow -1$ ), the scaling functions exhibit a high and narrow peak around  $y = 1$ . The position of the peak can be estimated as  $\langle y \rangle = \mu_2/(2\mu_1)$ , i.e., setting  $c = -1 + \varepsilon$ ,

$$\langle y \rangle^{(A)} = 1 + (3/2 - \gamma_E)\varepsilon + \dots, \quad \langle y \rangle^{(B)} = 1 + (2 - \gamma_E)\varepsilon + \dots, \quad (4.53)$$

whereas the squared width of the peak can be estimated as  $\text{var } y = (4\mu_1\mu_3 - 3\mu_2^2)/(12\mu_1^2)$ , i.e.,

$$\begin{aligned} \text{var } y^{(A)} &= \frac{5}{6}\varepsilon + \frac{19 - 10\gamma_E - \pi^2}{6}\varepsilon^2 + \dots, \\ \text{var } y^{(B)} &= \frac{2}{3}\varepsilon + \frac{22 - 8\gamma_E - \pi^2}{6}\varepsilon^2 + \dots, \end{aligned} \quad (4.54)$$

and finally the area under the peak scales as  $\mu_1$ , i.e.,

$$\mu_1^{(A)} = \frac{1}{\varepsilon} + 2 - \gamma_E + \dots, \quad \mu_1^{(B)} = \frac{1}{\varepsilon} + 3 - \gamma_E + \dots \quad (4.55)$$

We are thus left with the picture of a narrow peak around  $y = 1$ , whose width shrinks as  $\varepsilon^{1/2}$  and whose height grows as  $\varepsilon^{-3/2}$ .

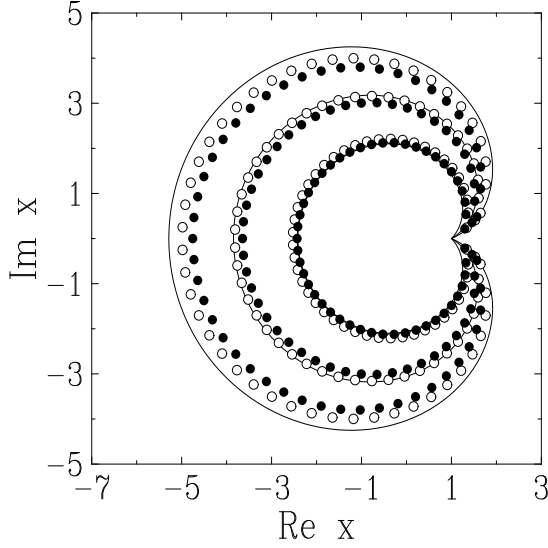
Let us close up this section with the location of the complex zeros of the polynomials  $F_n(x)$ . The derivation of the estimate (3.50) can be generalized to the present situation for arbitrary values of  $c$ . We are thus left with

$$\hat{F}_{n,\text{sing}}(u) \sim \left(1 - \frac{1}{(-u)^{c+2}}\right)^{-n}, \quad (4.56)$$

again with exponential accuracy. The asymptotic locus of the complex zeros is therefore given by the condition  $|1 - 1/(-u)^{c+2}| = 1$ . The relevant part of this locus can be parametrized by an angle  $0 \leq \theta \leq 2\pi$  as

$$u = (1 - e^{-i\theta})^{-1/(c+2)}, \quad x = \frac{1}{1 - (1 - e^{-i\theta})^{1/(c+2)}}. \quad (4.57)$$

This closed curve in the  $x$ -plane has a cusp at the point  $x = 1$ , corresponding to the scaling regime, with an opening angle equal to  $\pi/(c+2)$ . We have indeed



**Fig. 4.2** Plot of the non-trivial zeros of the polynomials  $F_n(x)$  in the complex  $x$ -plane. Symbols: zeros for  $n = 50$  in Case A (empty symbols) and Case B (full symbols). Lines: limiting curves with equation (4.57). From the inside to the outside:  $c = 0$  (the BA model, already shown in Figure 3.2),  $c = 1$  and  $c = 2$ .

$x - 1 \approx (e^{i\pi/2}\theta)^{1/(c+2)}$  as  $\theta \rightarrow 0$ . Figure 4.2 illustrates this result with data at time  $n = 50$  for three values of  $c$  and both initial conditions.

The exponential estimate (4.56) can again be recast into a large-deviation estimate for the probabilities  $f_k(n)$  in the regime  $k \sim n$ , of the form

$$f_k(n) \sim \exp(-nS(\zeta)), \quad (4.58)$$

with  $\zeta = k/n$ . The large-deviation function  $S(\zeta)$  is obtained in parametric form:

$$\zeta = \frac{(c+2)(v-1)}{v^{c+2}-1}, \quad (4.59)$$

$$S = \ln(v^{c+2}-1) - \frac{c+2}{v^{c+2}-1} \left( (v-1)\ln(v-1) + (v^{c+1}-1)v\ln v \right),$$

where the parameter  $v$  in the range  $1 < v < \infty$  is the opposite of the saddle-point value of  $u$  in the contour integral generalizing (3.52).

The power-law behavior

$$S(\zeta) \approx (c+1) \left( \frac{\zeta}{c+2} \right)^{\frac{c+2}{c+1}} \quad (4.60)$$

at small  $\zeta$  (corresponding to  $v \rightarrow \infty$ ) exactly matches the superexponential decay (4.46), (4.47) of the finite-size scaling functions  $\Phi^{(A)}(y)$  and  $\Phi^{(B)}(y)$ . The maximal value

$$S(1) = \ln(c+2), \quad (4.61)$$

corresponding to  $v \rightarrow 1$ , describes the exponential decay  $f_k(n) \sim (c+2)^{-n}$  of the probability of having a degree  $k$  equal to its maximal value ( $k = n$  or  $k = n-1$ ).

## 5 Discussion

In this paper we have presented a comprehensive study of finite-size (i.e., finite-time) effects on the degree statistics in growing networks. We have considered models defined by stochastic attachment rules, where nodes enter the network one at a time and attach to one single earlier node, so that the network has the topology of a tree. The present study thus generalizes and extends many results of References [11, 13, 14, 15, 16, 17, 18].

We have successively investigated the uniform attachment rule (UA), the linear attachment rule of the Barabási-Albert (BA) model, and a general preferential attachment rule (GPA) characterized by a continuous parameter  $c > -1$ , representing the initial attractiveness of a node. The UA and BA models are recovered as two special cases, respectively corresponding to  $c \rightarrow \infty$  and  $c = 0$ . The model is scalefree for any finite value of  $c$ , with the continuously varying exponents  $\gamma = c + 3$  and  $\nu = 1/(c + 2)$ . The continuous dependence of exponents on the parameter  $c$ , and the dependence of finite-size scaling functions on the initial condition (Case A or Case B in the present study), are two illustrations of the lack of universality which altogether characterizes the scaling behavior of growing networks.

The GPA rule is actually the most general one for which the partition function  $Z(n)$  (see (4.2)) is deterministic, i.e., independent of the history of the network. Whenever the attachment probability has a non-linear dependence on the degree  $k$ , the partition function becomes a history-dependent fluctuating quantity, so that the analysis of size effects becomes far more difficult. The general case of an arbitrary attachment rule, growing either less or more rapidly than linearly with the degree, has been considered in several works [14, 15, 18]. Whenever the degree dependence of the attachment rule is asymptotically linear, the resulting network is generically scalefree. The determination of the degree exponent  $\gamma$  is however a highly non-trivial task in general (see [14, 15] for an explicit example).

The present study has underlined the key rôle played by the typical value  $k_*(n)$  of the largest degree in a finite network at time  $n$ . In the UA model,  $k_*(n)$  grows logarithmically with time  $n$ . The situation is more interesting in the scalefree case, i.e., for  $c$  finite. The largest degree  $k_*(n)$  grows as a subextensive power law with exponent  $\nu$ , and demarcates three regimes in the size-degree plane, where finite-size (i.e., finite-time) effects on the degree distribution  $f_k(n)$  have different forms.

- In the stationary regime ( $k \ll k_*(n)$ ), the degree distribution is very close to the stationary one,  $f_{k,\text{stat}}$ .
- In the finite-size scaling regime ( $k \sim k_*(n)$ ), the degree distribution obeys a multiplicative finite-size scaling law. As already noticed in several earlier

works, the finite-size scaling function  $\Phi$  depends on the initial condition imposed on the network. This lack of universality holds for all finite values of the parameter  $c$ . Another feature of the finite-size scaling function is that it increases from its initial value  $\Phi(0) = 1$ , reaches a maximum, and stays above unity for a range of values of its argument  $y = k/k_*(n)$ , before it eventually falls off to zero. This non-monotonic overshooting behavior is however not mandatory. In this respect it is worth recalling the example of the so-called zeta urn model [22,23,24]. This mean-field interacting particle system with multiple occupancies possesses a continuous condensation transition at a finite critical density. Its behavior right at the critical density shares a high amount of similarity with the present problem, including a power-law stationary distribution with a continuously varying exponent, and finite-time scaling. The same results have been shown to apply to the dynamics of condensation in the zero-range process (ZRP) [25]. In the critical zeta urn and ZRP models, the finite-size scaling function is a monotonically decreasing function, so that  $\Phi(y) < 1$  for all  $y$ . This does not contradict the conservation of probability: the excess probability is carried by smaller values of  $k$ , pertaining to the stationary regime.

– In the large-deviation regime ( $k_*(n) \ll k \sim n$ ), the degree distribution falls off exponentially in  $n$ . At variance with the finite-size scaling law, the corresponding large-deviation function is independent of the initial condition. The analysis of this regime has been shown to be closely related to the locus of the complex zeros of the generating polynomials  $F_n(x)$ , which have played a central rôle throughout this work.

To close up, it is to be hoped that some of the concepts and methods used in the present work can be used to shed some new light either to other observables in the network models considered here, such as e.g. the statistics of leaders and lead changes [21], or to the degree statistics in more complex network models, such as e.g. the Bianconi-Barabási (BB) model [26,27], where attachment rules involve the competing effects of dynamical variables (the node degrees) and quenched disordered ones (the node fitnesses). Depending on the a priori distribution of the random fitnesses, the BB model may possess a low-temperature condensed phase. Some features of the dynamics of the condensed phase have been investigated recently, both at zero temperature [28], where the model is intimately related to the statistics of records, and at finite temperature [29].

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